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1+1 DIMENSIONAL YANG-MILLS THEORIES IN LIGHT-CONE GAUGE*

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In 1+1 dimensions two different formulations exist of SU(N) Yang Mills theories in light-cone gauge; only one of them gives results which comply with the ones obtained in Feynman gauge. Moreover the theory, when considered in 1+(D-1) dimensions, looks discontinuous in the limit D=2. All those features are proven in Wilson loop calculations as well as in the study of the $q\bar{q}$ bound state integral equation in the large N limit.

1. Introduction and motivations

In this report, we review some properties concerning Yang-Mills (YM) theories in 1+1 dimensions in the light-cone gauge. The reason why YM theories in 1+1 dimensions are interesting is at least twofold:

- The reduction of the dimensions to $D = 2$ entails tremendous simplifications in the theory, so that several important problems can be faced in this lower dimensional context. We are thinking for instance to the exact (when possible) evaluation of vacuum to vacuum amplitudes of Wilson loop operators, that, for a suitable choice of contour and in some specific limit, provide the potential between two static quarks. Another example is the spectrum of the Bethe-Salpeter equation, when dynamical fermions are added to the system.
- The second reason is that YM theories in $D = 2$ have several peculiar features that are interesting by their own. The most remarkable ones are:
  a) in $D = 2$ within the same gauge choice (light-cone gauge) two inequivalent formulations of the theory seem to coexist;
  b) $D = 2$ is a point of discontinuity for YM theories; this is an intriguing feature whose meaning has not been fully understood so far.

All the features we have listed are most conveniently studied if the light-cone gauge (lcg) is chosen. In such a gauge the Faddeev Popov sector decouples and the unphysical degrees of freedom content of the theory is minimal. The price to be paid

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for these nice features is the presence of the so called 'spurious' poles in the vector propagator.[1] In fact, in the gauge $nA = 0$ with n^μ a given constant null vector ($n^2 = 0$), the form of the propagator in D dimensions turns out to be

$$D^{ab}_{\mu\nu}(k) = \frac{-i\delta^{ab}}{k^2 + i\epsilon}\left(g_{\mu\nu} - \frac{n_\mu k_\nu + n_\nu k_\mu}{nk}\right) . \tag{1}$$

As we shall see, to handle the spurious pole at $nk = 0$ is a delicate matter; basically all difficulties encountered in the past within the lcg quantization are related to this problem.

In Sect. 2 we focus on the $D \neq 2$ case, and discuss the so called 'manifestly unitary' and 'causal' formulations of the theory. We shall see that the correct formulation is the causal one: the manifestly unitary formulation will meet so many inconsistencies to make it unacceptable. Moreover, even in the causal formulation, the theory looks discontinuous in the limit $D = 2$.

In Sect. 3 we compare the two formulations at strictly $D = 2$. Surprisingly, in this case both seem to coexist, without obvious inconsistencies. Thus, a natural question arises: are the two quantization schemes equivalent in $D = 2$? Do they provide us with equal results?

The answer to these questions are given in Sect. 4 where Wilson loop expectation values are evaluated. We shall find that the two formulations are indeed inequivalent.

In Sect. 5 the theory will be considered on the cylinder $\mathcal{R} \times \mathcal{S}$, namely with the space variable constrained in an interval, in order to reach a consistent infrared (IR) regularization. Again the two formulations behave quite differently.

Finally Sect. 6 contains a discussion of the bound state integral equation when dynamical fermions are present and our conclusions.

2. $D \neq 2$: a comparison between manifestly unitary and causal formulations

A manifestly unitary formulation of YM theories in lcg can be obtained by quantizing the theory in the so called null-frame formalism, i.e. passing in light-cone coordinates and interpreting x^+ as the evolution coordinate (time) of the system; the remaining components $x^-, x_\perp$ will be interpreted as 'space' coordinates. Within this quantization scheme, one of the unphysical components of the gauge potential (say A_-) is set equal to zero by the gauge choice whereas the remaining unphysical component (A_+) is no longer a dynamical variable but rather a Lagrange multiplier of the secondary constraint (Gauss' law). Thus, already at the classical level, it is possible to restrict to the phase space containing only the physical (transverse) polarization of the gauge fields. Then, canonical quantization on the null plane provides the answer to the prescription for the spurious pole in the propagator, the answer being essentially the Cauchy principal value (CPV) prescription.

Unfortunately, following this scheme, several inconsistencies arise, all of them being related to the violation of causality that CPV prescription entails:

- non-renormalizability of the theory: already at the one loop level, dimensionally regularized Feynman integrals develop loop singularities that appear as double poles at $D = 4$.[2]
- power counting criterion is lost: the pole structure in the complex k_0 plane is such that spurious poles contribute under Wick rotation. As a consequence euclidean Feynman integrals are not simply related to Minkowskian ones as an extra contribution shows up which jeopardizes naive power counting.[3]
- gauge invariance is lost: due to the above mentioned extra contributions, the $N = 4$ supersymmetric version of the theory turns out not to be finite, at variance with the Feynman gauge result.[2]

Consequently, manifestly unitary theories do not seem to exist. As explained above, all the bad features of this formulation have their root in the lack of causality of the prescription for the spurious pole, and the subsequent failure of the power counting criterion for convergence. Thus, a natural way to circumvent these problems is to choose a causal prescription. It was precisely following these arguments that Mandelstam and Leibbrandt [4], independently, introduced the ML prescription

$$\frac{1}{k_-} \equiv ML(\frac{1}{k_-}) = \frac{k_+}{k_+k_- + i\epsilon} = \frac{1}{k_- + i\epsilon \mathrm{sign}(k_+)}\,. \tag{2}$$

It can be easily realized that with this choice the position of the spurious pole is always ‘coherent’ with that of Feynman ones, no extra terms appearing after Wick rotation which threaten the power counting criterion for convergence. How can one justify such a recipe? One year later Bassetto and collaborators [5] filled the gap by showing that ML prescription arises naturally by quantizing the theory at equal time, rather than at equal x^+. Eventually they succeeded [6] in proving full renormalizability of the theory and full agreement with Feynman gauge results in perturbative calculations.[7]

At present the level of accuracy of the light-cone gauge is indeed comparable with that of the covariant gauges.

An important point to be stressed is that equal time canonical quantization in lcg, leading to the ML prescription for the spurious pole, does not provide us with a manifestly unitary formulation of the theory. In fact in this formalism Gauss’ laws do not hold strongly but, rather, the Gauss’ operators obey to a free field equation and entail the presence in the Fock space of unphysical degrees of freedom. The causal nature of the ML prescription for the spurious poles is a consequence of the causal propagation of those ‘ghosts’. A physical Hilbert space can be selected by imposing the (weakly) vanishing of Gauss’ operators. This mechanism is similar to the Gupta Bleuler quantization scheme for electrodynamics in Feynman gauge, but with the great advantage that it can be naturally extended to the non abelian case without Faddeev Popov ghosts.[1]

3. $D = 2$**: a comparison between the manifestly unitary and causal formulations**

The causal formulation of the theory can be straightforwardly extended to *any* dimension, including the case $D = 2$. On the other hand, the manifestly unitary formulation can *only* be defined in $D = 2$ without encountering obvious inconsistencies. The reason is simple: all problems were related to the lack of causality encoded in the CPV prescription. But at exactly $D = 2$ there are no physical degrees of freedom propagating at all, and then causality is no longer a concern. Moreover, at exactly $D = 2$ and within the lcg, the 3- and 4- gluon vertices vanish, so that all the inconsistencies related to the perturbative evaluation of Feynman integrals are no longer present in this case. A manifestly unitary formulation provides the following 'instantaneous - Coulomb type' form for the only non vanishing component of the propagator:

$$D^{ab}_{++}(x) = -\frac{i\delta^{ab}}{(2\pi)^2}\int d^2k\, e^{ikx}\,\frac{\partial}{\partial k_-}P\left(\frac{1}{k_-}\right) = -i\delta^{ab}\,\frac{|x^-|}{2}\,\delta(x^+)\,, \tag{3}$$

where P denotes CPV prescription, whereas equal time canonical quantization gives, for the same component of the propagator,

$$D^{ab}_{++}(x) = \frac{i\delta^{ab}}{\pi^2}\int d^2k\, e^{ikx}\,\frac{k_+^2}{(k^2+i\epsilon)^2} = \frac{\delta^{ab}(x^-)^2}{\pi(-x^2+i\epsilon)}\ . \tag{4}$$

Thus, it seems we have two different formulation of YM theories in $D = 2$, and within the same gauge choice, the lcg.[8] Whether they are equivalent and, in turn, whether they are equivalent to a different gauge choice, such as Feynman gauge, has to be explicitly verified.

We can summarize the situation according to the content of unphysical degrees of freedom. Since the paper by 't Hooft in 1974 [9], it is a common belief that pure YM in $D = 2$ is a theory with no propagating degrees of freedom. This happens in the manifestly unitary formulation leading to CPV prescription for the spurious pole and to the propagator (3). This formulation, however, cannot be extended outside $D = 2$ without inconsistencies. Alternatively, we have the same gauge choice but with a different quantization scheme, namely at equal time, leading to the causal (ML) prescription for the spurious pole and to the propagator (4). Here, even in the pure YM case, some degrees of freedom survive, as we have propagating ghosts. Such a formulation is in a better shape when compared to the previous one as it can be smoothly extended to any dimension, where consistency with Feynman gauge has been established.

Feynman gauge validity for any $D \neq 2$ is unquestionable, while, at strictly $D = 2$, the vector propagator in this gauge fails to be a tempered distribution. Still, in the spirit of dimensional regularization, one can always evaluate amplitudes in $D \neq 2$ and take eventually the limit $D \to 2$. In following this attitude, the number of degrees of freedom is even bigger as Faddeev-Popov ghosts are also to

be taken into account. In addition, in the covariant gauge 3- and 4- gluon vertices do not vanish and the theory does not look free at all.

4. Wilson loops calculations

To clarify the whole matter we need a test of gauge invariance. In particular, we want to answer the following three questions:

(i) Is YM theory continuous in the limit $D \to 2$?
(ii) Is YM theory in $D = 2$ a free theory?
(iii) Are the two lcg formulations in $D = 2$ equivalent?

To probe gauge invariance and to answer the above questions, following ref.[8], we shall evaluate vacuum to vacuum amplitudes of Wilson loop operators, defined as a functional of the closed contour γ through

$$W[\gamma] = \frac{1}{N} \int dA_\mu \delta(\Phi(A)) \det[M_\Phi] e^{i\int dx \mathcal{L}(x)} \,\mathrm{Tr}\left\{ \mathcal{P} e^{ig\oint_\gamma dx^\mu A^a_\mu T^a} \right\} , \tag{5}$$

where for convenience we choose $SU(N)$ as gauge group with hermitean generators T^a. In Eq.(5), $\Phi(A) = 0$ denotes the gauge choice and $\det[M_\Phi]$ the corresponding Faddeev-Popov determinant, that can be either trivial or not, depending on the gauge $\Phi(A)$. As usual, $\mathcal{P}$ denotes ordering along the closed path γ, that we shall choose to be a light-like rectangle in the plane (x^+, x^-) with length sides (T, L). For later convenience, we recall that the Casimir constants of the fundamental and adjoint representations, C_F and C_A, are defined through

$$C_F = (1/N)\mathrm{Tr}(T^a T^a) = (N^2 - 1)/2N\ , \quad \text{and} \quad C_A \delta^{ab} = f^{acd} f^{bcd} = \delta^{ab} N\ , \tag{6}$$

f^{abc} being the structure constants for $SU(N)$.

First of all we shall check continuity in the $D \to 2$ limit. To this purpose, we have to choose the lcg in its causal formulation. In fact, among the gauge choices we considered, this is the only one whose formulation is smooth in the $D \to 2$ limit.[a] Within this gauge choice, only a perturbative $\mathcal{O}(g^4)$ calculation is viable. Performing the calculation in D dimensions and eventually taking the limit $D \to 2$, the expression for the Wilson loop gives

$$\lim_{D\to 2} W^{(D)}_{ML}(\gamma) = 1 - i\frac{g^2}{2} LTC_F - \frac{g^4}{8}(LT)^2 \left[C_F^2 - \frac{C_F C_A}{8\pi^2}\left(1 + \frac{\pi^2}{3}\right)\right] + \mathcal{O}(g^6)\ , \tag{7}$$

whereas the same quantity evaluated at exactly $D = 2$ gives a different answer, namely

$$W^{(D=2)}_{ML}(\gamma) = 1 - i\frac{g^2}{2} LTC_F - \frac{g^4}{8}(LT)^2 \left[C_F^2 - \frac{C_F C_A}{24}\right] + \mathcal{O}(g^6)\ . \tag{8}$$

[a] In fact, lcg in its manifestly unitary formulation is acceptable only at $D = 2$ and therefore cannot be used to check continuity; even Feynman gauge cannot be used, as the propagator is divergent at $D = 2$, preventing therefore a calculation at exactly $D = 2$.

Thus, we have a surprising result: YM theories are discontinuous at $D = 2$. The technical reason of such discontinuity can be easily understood in terms of 'anomalous' diagrams that survive in the limit $D \to 2$. In strictly $D = 2$, as already stressed, the 3- and 4- gluon vertices vanish in lcg. Consequently, the free propagator (4) is the *complete* two point Green function, as there are no radiative corrections. On the other hand, in $D = 2 + \varepsilon$ the gluon vertices do not vanish anymore, as 'ε' transverse components couple the gauge fields. Thus, in $D \neq 2$ dimensions the two-point Green function has radiative corrections. The one loop correction, $\mathcal{O}(g^2)$, is the standard 'bubble diagram', with two free propagators connected by two 3-gluon vertices. Obviously, the strength of the vertices vanishes in the limit $\varepsilon = (D-2) \to 0$; nevertheless,this correction to the Green function produce a finite contribution in the limit $\varepsilon \to 0$ due to the matching with the loop pole precisely at $D = 2$. Such a dimensional 'anomaly-type' phenomenon is responsible of the discontinuity of YM theory at $D = 2$.

As a matter of fact, it is easy to evaluate the contribution to the Wilson loop given by this anomalous part of Green function, surviving in the limit $D \to 2$: it provides the factor $g^4(LT)^2 C_F C_A/64\pi^2$, which is indeed the difference between Eq. (7) and Eq. (8). This discontinuity at $D = 2$ is a very interesting phenomenon, whose nature is still unclear; whether it is related to a true anomaly, *i.e.* whether there is a classical symmetry violated at the quantum level, is still a matter of investigation.

Consistency with Feynman gauge can be checked by evaluating the dimensionally regularized Wilson loop: an $\mathcal{O}(g^4)$ calculation provides exactly the same result of lcg in its causal formulation for any D and therefore also in the limit $D \to 2$: as expected, we have full agreement between Feynman and light-cone gauge if the ML prescription for the spurious poles is adopted. We stress that the 'anomalous' self-energy contribution we have hitherto discussed, is essential in order to get such an agreement.[7]

However, both in the dimensionally regularized case with the limit $D \to 2$ taken at the end and in the strictly $D = 2$ case, within the causal lcg formulation, we realize that the Wilson loop results do not depend only on C_F: a 'genuine' non abelian $C_F C_A$ dependence appears at $\mathcal{O}(g^4)$. This means that, although the vertices vanish, YM in $D = 2$ dimensions is *not* equivalent to an abelian theory.

On the contrary, this feature does not occur in the manifestly unitary (strictly 2-dimensional) formulation. In fact, due to the contact nature of the propagator, Eq. (3), non-abelian C_A-dependent terms do not appear in the expression of Wilson loops. In this case, it is easy to find that the perturbative result exponentiates in a simple abelian way

$$W_{CPV}^{(D=2)}(\gamma) = e^{-ig^2 LTC_F/2} \, . \tag{9}$$

Pure YM in its manifestly unitary formulation is essentially free and equivalent to an abelian theory. We are lead to conclude that the two light-cone formulations at $D = 2$ are indeed inequivalent.

Summarizing, three different evaluations of the same Wilson loop within the

same gauge choice (lcg) provided us with three different answers! Discrepancy between Eqs. (8) and (9) is explained by the coexistence of two different inequivalent formulations of YM theory, whereas discrepancy between Eqs. (7) and (8) is explained by the discontinuity in the limit $D \to 2$.

However, in all the cases we considered, we always got at least a pure 'area-law' dependence of the Wilson loop. Is this an universal property of $D = 2$ YM theory? Contrary to a common belief, we shall show that this is not the case, by providing an explicit counterexample.[10]

Let us consider again a rectangular loop $\tilde{\gamma}$ with area $A = LT$, but now centered at the origin of the plane (x^0, x^1). For convenience, let us stick to the $D = 2$ case and focus on the two different formulations of lcg. From a physical point of view, this contour is even more interesting: would one be able to compute the exact value of the Wilson loop amplitude, one could derive the potential $V(L)$ between two static quarks separated by a distance L through the well known formula

$$\lim_{T\to\infty} W(\tilde{\gamma}) = e^{-iTV(L)} \tag{10}$$

In the manifestly unitary (CPV) case, due to the contact nature of the potential (3), the Wilson loop can again be exactly evaluated giving, for a finite size of the rectangle,

$$W^{(D=2)}_{CPV}(\tilde{\gamma}) = e^{-ig^2 C_F LT/2} \tag{11}$$

and therefrom a linear confining potential between quarks with string tension $\sigma = g^2 C_F/2$. However, it should be emphasized that such a confining result for QCD_2 has the same origin as in QED, namely follows from the *abelian* 'contact' nature of the potential. In the causal (ML) case a complete evaluation at all orders is not viable due to the presence of genuine non abelian terms. Only a perturbative $\mathcal{O}(g^4)$ evaluation is viable and after lengthy calculations, one finds

$$\begin{aligned} W^{(D=2)}_{ML}(\tilde{\gamma}) \quad = \quad & 1 - i\frac{g^2}{2}LTC_F - \frac{g^4}{8}(LT)^2 \left\{ C_F^2 + \frac{C_F C_A}{4\pi^2}\left[3 + \frac{2\pi^2}{3} + 2\beta[1+ \right.\right. \\ & (2+\beta)\ln\beta] - 2(1+\beta)^2\ln(1+\beta) - \frac{2}{3\beta}\ln^2(1+\beta) - \frac{1}{6\beta^2}\times \\ & (1-\beta)^2\ln^2(1-b) - \frac{1}{3\beta^2}(1-\beta)^4\left(\mathrm{Li}(\beta) + \mathrm{Li}\left(-\frac{\beta}{1+\beta}\right)\right) - \\ & \left.\left.\frac{1}{\beta}\left(\mathrm{Li}(\beta) + \mathrm{Li}\left(\frac{\beta}{1+\beta}\right)\right)\right]\right\} + \mathcal{O}(g^6) \ . \end{aligned} \tag{12}$$

The Wilson loop amplitude, for finite L and T, not only depends on the area, but also on the dimensionless ratio $\beta = L/T$ through a complicated factor involving the dilogarithm function $\mathrm{Li}(z)$.

Obviously, the fact that in this case we only have a perturbative $\mathcal{O}(g^4)$ calculation prevent us from making any interpretation of the result in term of a potential between static quarks in the large T limit. Nevertheless, it is remarkable and perhaps not incidental that in such a limit all the dependence on β cancel leaving again

a pure area dependence

$$\lim_{T\to\infty} W_{ML}^{(D=2)}(\tilde{\gamma}) = 1 - i\frac{g^2}{2}LTC_F - \frac{g^4}{8}(LT)^2\left\{C_F^2 + \frac{C_F C_A}{12\pi^2}(9+2\pi^2)\right\} \quad (13)$$

with finite coefficients. We stress that again the same theory with the same gauge choice leads to *different* results when using different expressions for the two point Green function, even in the large T limit.

5. Wilson loops on the cylinder

While a comparison with Feynman gauge at $D \neq 2$ gave a satisfactory result, a comparison at strictly $D = 2$ is impossible owing to the well-known IR singular behaviour of the vector propagator in Feynman gauge. Then, in order to achieve a consistent IR regularization, we consider the theory on the cylinder $\mathcal{R} \times \mathcal{S}$, namely we restrict the space variable to the interval $-L \leq x \leq L$ with periodic boundary conditions on the potentials. Time is *not* compactified. We follow here the treatment given in ref.[11].

In so doing new features appear owing to the non trivial topology of the cylinder and we feel preliminary to examine the equal-time quantization of the pure YM theory in the light-cone gauge $A_- = 0$. Introduction of fermions at this stage would not entail particular difficulties, but would be inessential to our subsequent argument.

We recall that axial-type gauges cannot be defined on compact manifolds without introducing singularities in the vector potentials (Singer's theorem).[12] Partial compactifications are possible provided they occur in a direction different from the one of the gauge fixing vector: this is indeed what happens in the present case.

Starting from the standard lagrangian density (for SU(N))

$$\mathcal{L} = -1/2\, Tr(F^{\mu\nu}F_{\mu\nu}) - 2Tr(\lambda n A), \quad (14)$$

$n_\mu = \frac{1}{\sqrt{2}}(1,1)$ being the gauge vector and λ being Lagrange multipliers, which actually coincide with Gauss' operators, it is straightforward to derive the equations of motion

$$\begin{aligned} &A_- = 0, \;\; \partial_-{}^2 A_+ = 0, \\ &\partial_-\partial_+ A + - ig[A_+, \partial_- A_+] = \lambda. \end{aligned} \quad (15)$$

As a consequence we get

$$\partial_-\lambda = 0. \quad (16)$$

In a 'light-front' treatment (quantization at equal x^+), this equation would be a constraint and ∂_- might be inverted (with suitable boundary conditions) to get the 'strong' Gauss' laws

$$\lambda = 0. \quad (17)$$

This would correspond in the continuum to the CPV prescription for the singularity at $k_- = 0$ in the relevant Green functions.

In equal-time quantization eq.(16) is an evolution equation. The Gauss' operators do not vanish strongly: Gauss' laws are imposed as conditions on the 'physical' states of the theory. In so doing one can show [11] that the only surviving 'physical' degrees of freedom are zero modes of the potentials related to phase factors of contours winding around the cylinder. Frequency parts are unphysical, but non vanishing: they contribute indeed to the causal expression of the vector propagator

$$\begin{aligned} G_c(t,x) &= G(t,x) - \frac{it}{4L} Pctg\big(\frac{\pi\sqrt{2}x^+}{2L}\big), \\ G(t,x) &= 1/2\,|t|\,\big(\delta_p(x+t) - \frac{1}{2L}\big), \end{aligned} \tag{18}$$

δ_p being the periodic generalization of the Dirac distribution.

G_c looks like a complex "potential" kernel, the absorptive part being related to the presence of ghost-like excitations, which are essential to recover the ML prescription in the decompactification limit $L \to \infty$; as a matter of fact in this limit $G(t,x)$ becomes the 'instantaneous' 't Hooft potential, whereas G_c is turned into the causal ML distribution.

We are now in the position of comparing a Wilson loop on the cylinder when evaluated according to the 't Hooft potential or using the causal light-cone propagator.

In order to avoid an immediate interplay with topological features, we consider a Wilson loop entirely contained in the basic interval $-L \le x \le L$. We choose again a rectangular Wilson loop γ with light–like sides, directed along the vectors n_μ and n^*_μ, with lengths λ and τ respectively, and parametrized according to the equations:

$$\begin{aligned} C_1 : x^\mu(s) &= n^\mu \lambda s, \\ C_2 : x^\mu(s) &= n^\mu \lambda + n^{*\mu}\tau s, \\ C_3 : x^\mu(s) &= n^{*\mu}\tau + n^\mu \lambda(1-s), \\ C_4 : x^\mu(s) &= n^{*\mu}\tau(1-s), \qquad 0 \le s \le 1, \end{aligned} \tag{19}$$

with $\lambda + \tau < 2\sqrt{2}L$. We are again interested in the quantity

$$W(\gamma) = \frac{1}{N}\Big\langle \mathbf{0} | Tr \mathcal{T}\mathcal{P}\Big(exp\Big[ig \oint_\gamma A dx^+ \Big]\Big) | \mathbf{0} \Big\rangle, \tag{20}$$

where $\mathcal{T}$ means time-ordering and $\mathcal{P}$ color path-ordering along γ.

The vacuum state belongs to the physical Hilbert space as far as the non vanishing frequency parts are concerned; it is indeed the Fock vacuum $|\mathbf{\Omega}\Big\rangle$. Then we consider its direct product with the lowest eigenstate of the Hamiltonian concerning zero modes (see [11]). Due to the occurrence of zero modes, we cannot define a "bona fide" complete propagator for our theory: on the other hand a propagator is not required in eq.(20).

We shall first discuss the simpler case of QED, where no color ordering is involved. Eq.(20) then becomes

$$W(\gamma) = \Big\langle \mathbf{0}|\mathcal{T}\Big(exp\Big[ig\oint_\gamma A dx^+\Big]\Big)|\mathbf{0}\Big\rangle, \tag{21}$$

and a little thought is enough to realize the factorization property

$$\begin{aligned} W(\gamma) &= \Big\langle \mathbf{0}|\mathcal{T}\Big(exp\Big[\frac{ig}{\sqrt{2L}}\oint_\gamma (b_0 + a_0 t)dx^+\Big]\Big)|\mathbf{0}\Big\rangle\Big\langle \mathbf{0}|\mathcal{T}\Big(exp\Big[ig\oint_\gamma \hat{A}(t,x)dx^+\Big]\Big)|\mathbf{0}\Big\rangle \\ &= W_0 \cdot \hat{W}, \end{aligned} \tag{22}$$

according to the splitting of the potential in zero mode and frequency parts. In turn the Wilson loop $\hat{W}$ can also be expressed as a Feynman integral starting from the QED lagrangian , without the zero mode

$$\hat{W}(\gamma) = \mathcal{N}^{-1}\Big(exp\Big[g\oint_\gamma \frac{\partial}{\partial J}dx^+\Big]\Big)\Big[\int \mathcal{D}\hat{A}\,\mathcal{D}\lambda\, exp\; i\Big(\int d^2x(\mathcal{L}+J\hat{A})\Big)\Big]\Big|_{J=0}, \tag{23}$$

$\mathcal{N}$ being a suitable normalization factor.

Standard functional integration gives

$$\begin{aligned} \hat{W}(\gamma) &= \mathcal{N}^{-1}\Big(exp\Big[g\oint_\gamma \frac{\partial}{\partial J}dx^+\Big]\Big) \\ & \quad exp\Big[\frac{i}{2}\iint d^2\xi d^2\eta J(\xi)G_c(\xi-\eta)J(\eta)\Big]\Big|_{J=0}. \end{aligned} \tag{24}$$

and we are led to the expression

$$\begin{aligned} \hat{W}(\gamma) &= exp\Big[i\,g^2\oint_\gamma dx^+\oint_\gamma dy^+\, G_c(x^+ - y^+, x^- - y^-)\Big] \\ &= exp\Big[i\,g^2\oint_\gamma dx^+\oint_\gamma dy^+\, G(x^+ - y^+, x^- - y^-)\Big] \\ &= exp\Big[-i\frac{g^2\mathcal{A}}{2}\Big]\,exp\Big[-i\,g^2\oint_\gamma dx^+\oint_\gamma dy^+\,\frac{|x^+ + x^- - y^+ - y^-|}{4L\sqrt{2}}\Big] \end{aligned} \tag{25}$$

the absorptive part of the "potential" averaging to zero in the abelian case. Therefore the abelian Wilson-loop calculation is unable to discriminate between the two different Green functions G_c and G. The quantity $\mathcal{A} = \lambda\tau$ is the area of the loop. The same result can also be obtained by operatorial techniques, using Wick's theorem and the canonical algebra.

We are thereby left with the problem of computing W_0. In [11] we have shown that the zero mode contribution *exactly cancels* the last exponential in eq.(25), leaving the pure loop area result, only as a consequence of the canonical algebra, and even

in the presence of a topological degree of freedom. The result coincides with the one we would have obtained introducing in eq. (25) the complete Green's function, *i.e.* with the zero mode included. The same area result is obtained also in the non abelian case if we use the 't Hooft form for the propagator $\frac{1}{2}|t|\delta_p(x+t)$, in spite of the fact that this form has not a sound canonical basis and that factorization in (22) is no longer justified in the non abelian case. As a matter of fact a little thought is enough to realize that only planar diagrams survive thanks to the 'contact' nature of the potential, leading to the expression

$$W(\gamma) = exp\Big[- i\,\frac{g^2\, C_F\, \lambda\, \tau}{2}\Big]. \tag{26}$$

The area ($\mathcal{A} = \lambda\, \tau$) law behaviour of the Wilson loop we have found in this case together with the occurrence of a simple exponentiation in terms of the Casimir of the fundamental representation, is a quite peculiar result, insensitive to the decompactification limit $L \to \infty$. It is rooted in the particularly simple expression for the "potential" we have used that coincides with the one often considered in analogous Euclidean calculations [13].

However canonical quantization suggests that we should rather use the propagator $G_c(t, x)$. A full resummation of perturbative exchanges is no longer viable in this case, owing to the presence of non vanishing cross diagrams, in which topological excitations mix non trivially with the frequency parts. Already at $\mathcal{O}(g^4)$, a tedious but straightforward calculation of the sum of all the "cross" diagrams leads to the result

$$W_{cr} = -(\frac{g^2}{4\pi})^2\, 4\, C_F\, (C_F - \frac{C_A}{2})(\mathcal{A})^2 \int_0^1 d\,\xi \int_0^1 d\,\eta\, log \frac{|sin\rho(\xi-\eta)|}{|sin\rho\xi|} log \frac{|sin\rho(\xi-\eta)|}{|sin\rho\eta|}, \tag{27}$$

where $\rho = \frac{\pi\lambda}{\sqrt{2}L}$. One immediately recognizes the appearance of the quadratic Casimir of the adjoint representation (C_A); moreover, a dimensionless parameter ρ, which measures the ratio of the side length λ to the interval length L, explicitly occurs. In the decompactification limit $\rho \to 0$, the expression of the cross graph given in ref. [8]

$$W_{cr} = -(\frac{g^2}{4\pi})^2\, 2C_F\, (C_F - C_A/2)(\mathcal{A})^2 \frac{\pi^2}{3} \tag{28}$$

is smoothly recovered. Of course the finite self-energy contribution found in ref. [8] in the dimensionally regularized theory cannot appear in a strictly 1+1 dimensional treatment. It is perhaps not surprising that in the limit $L \to \infty$ the perturbative result for W_{cr} in the continuum is correctly reproduced, in spite of the presence of topological excitations. Still the difference between the result obtained with the 'contact' potential and with the causal one is even more striking: in both cases at large N only planar diagrams survive but they nevertheless give rise to different expressions for the same Wilson loop.

It seems that *planarity* is not enough to single out an unambiguous result.

6. The 't Hooft bound state equation

In 1974 G. 't Hooft [9] proposed a very interesting model to describe the mesons, starting from a SU(N) Yang-Mills theory in 1+1 dimensions in the large N limit.

Quite remarkably in this model quarks look confined, while a discrete set of quark-antiquark bound states emerges, with squared masses lying on rising Regge trajectories.

The model is solvable thanks to the "instantaneous" character of the potential acting between quark and antiquark.

Three years later such an approach was criticized by T.T. Wu [14], who replaced the instantaneous 't Hooft's potential by an expression with milder analytical properties, allowing for a Wick's rotation without extra terms.

Unfortunately this modified formulation led to a quite involved bound state equation, which may not be solved. An attempt to treat it numerically in the zero bare mass case for quarks [15] led only to partial answers in the form of a completely different physical scenario. In particular no rising Regge trajectories were found.

After those pioneering investigations, many interesting papers followed 't Hooft's approach, pointing out further remarkable properties of his theory and blooming into the recent achievements of two dimensional QCD , whereas Wu's approach sank into oblivion.

Still, equal time canonical quantization of Yang-Mills theories in light-cone gauge [5] leads precisely in 1+1 dimensions to the Wu's expression for the vector exchange between quarks [8], which is nothing but the 1+1 dimensional version of the Mandelstam-Leibbrandt (ML) propagator. We have already stressed that this option is mandatory in order to achieve gauge invariance and renormalization in 1+(D-1) dimensions.

We follow here definitions and notations of refs.[9] and [14] the reader is invited to consult.

The 't Hooft potential exhibits an infrared singularity which, in the original formulation, was handled by introducing an infrared cutoff; a quite remarkable feature of this theory is that bound state wave functions and related eigenvalues turn out to be cutoff independent. As a matter of fact in ref. [16], it has been pointed out that the singularity at $k_- = 0$ can also be regularized by a Cauchy principal value (CPV) prescription without finding differences in gauge invariant quantities. Then, the difference between the two potentials is represented by the following distribution

$$\Delta(k) \equiv \frac{1}{(k_- - i\epsilon sign(k_+))^2} - P\Big(\frac{1}{k_-^2}\Big) = -i\pi sign(k_+)\delta'(k_-). \tag{29}$$

In ref.[17], which we closely follow in the sequel, this quantity has been treated as an insertion in the Wu's integral equations for the quark propagator and for the bound state wave function, starting from 't Hooft's solutions. *Exactly* the same

planar diagrams of refs.[9] and [14], which are the relevant ones in the large N limit, are summed.

The Wu's integral equation for the quark self-energy in the Minkowski momentum space is

$$\begin{aligned}\Sigma(p;\eta) &= i\frac{g^2}{\pi^2}\frac{\partial}{\partial p_-}\int dk_+dk_-\Big[P\Big(\frac{1}{k_- - p_-}\Big) + i\eta\pi sign(k_+ - p_+)\delta(k_- - p_-)\Big] \\ &\cdot \frac{k_-}{k^2 + m^2 - k_-\Sigma(k;\eta) - i\epsilon},\end{aligned} \tag{30}$$

where $g^2 = g_0^2\, N$ and η is a real parameter which is used as a counter of insertions and eventually should be set equal to 1.

Its exact solution with appropriate boundary conditions reads

$$\begin{aligned}\Sigma(p;\eta) &= \frac{1}{2p_-}\Big(\Big[p^2 + m^2 + (1-\eta)\frac{g^2}{\pi}\Big] - \Big[p^2 + m^2 - (1-\eta)\frac{g^2}{\pi}\Big] \\ &\cdot \sqrt{1 - \frac{4\eta g^2 p^2}{\pi(p^2 + m^2 - (1-\eta)\frac{g^2}{\pi} - i\epsilon)^2}}\ \Big).\end{aligned} \tag{31}$$

One can immediately realize that 't Hooft's and Wu's solutions are recovered for $\eta = 0$ and $\eta = 1$ respectively.

The dressed quark propagator turns out to be

$$S(p;\eta) = -\frac{ip_-}{m^2 + 2p_+p_- - p_-\Sigma(p;\eta)}. \tag{32}$$

Wu's bound state equation in Minkowski space, using light-cone coordinates, is

$$\begin{aligned}\psi(p,r) &= \frac{-ig^2}{\pi^2}S(p;\eta)S(p-r;\eta)\int dk_+dk_-\Big[P\Big(\frac{1}{(k_- - p_-)^2}\Big) - \\ &- i\eta\pi sign(k_+ - p_+)\delta'(k_- - p_-)\Big]\psi(k,r).\end{aligned} \tag{33}$$

We are here considering for simplicity the equal mass case and η should be set equal to 1.

Let us denote by $\phi_k(x),\ 0 \le x = \frac{p_-}{r_-} \le 1,\ r_- > 0$, the 't Hooft's eigenfunction corresponding to the eigenvalue α_k for the quantity $\frac{-2r_+r_-}{M^2}$, where $M^2 = m^2 - \frac{g^2}{\pi}$. Those eigenfunctions are real, of definite parity under the exchange $x \to 1-x$ and vanishing outside the interval $0 < x < 1$:

$$\begin{aligned}\phi_k(x) &= \int dp_+\frac{r_-}{M^2}\psi_k(p_+, p_-, r), \\ i\pi\,\psi_k &= \phi_k(x)\frac{M^4}{M^2 + 2r_-p_+x - i\epsilon}\,\cdot \\ &\cdot\ \frac{1 - \alpha_k x(1-x)}{M^2 - \alpha_k M^2(1-x) - 2r_-p_+(1-x) - i\epsilon}.\end{aligned} \tag{34}$$

They are solutions of eq.(33) for $\eta = 0$ and form a complete set.

We are interested in a first order calculation in η. This procedure is likely to be sensible only in the weak coupling region $\frac{g_0^2}{\pi} < m^2$. The integral equation (33), after first order expansion in η of its kernel, becomes

$$
\begin{aligned}
\psi(p_+,p_-,r) &= \frac{ig^2}{\pi^2}\frac{p_-}{M^2+2p_+p_- - i\epsilon}\frac{p_- - r_-}{M^2+2(p_+-r_+)(p_- - r_-)-i\epsilon} \\
\cdot\Big[\Big(1-\frac{\eta g^2 M^2}{\pi} &\quad [\quad (M^2+2p_+p_- - i\epsilon)^{-2} + (M^2+2(p_+-r_+)(p_- - r_-)-i\epsilon)^{-2}]\Big) \\
&\cdot \quad \int dk_+ dk_- P\frac{1}{(k_- - p_-)^2}\psi(k_+,k_-,r) - \\
&- \quad i\pi\eta\int dk_+dk_- sign(k_+ - p_+)\delta'(k_- - p_-)\psi(k_+,k_-,r)\Big]. \qquad (35)
\end{aligned}
$$

We integrate this equation over p_+ with $r_- > 0$ and look for solutions with the same support properties of 't Hooft's ones. We get

$$
\begin{aligned}
\phi(x,r) &= \frac{g^2}{\pi M^2}\frac{x(1-x)}{1-\alpha x(1-x)-i\epsilon}\Big[\Big(1-\eta\frac{g^2}{\pi M^2}\frac{x^2+(1-x)^2}{(1-\alpha x(1-x)-i\epsilon)^2}\Big) \\
\cdot \quad P &\int_0^1 \frac{dy}{(y-x)^2}\phi(y,r) - \frac{\alpha\eta}{2}\int d\xi \log\frac{\frac{1}{1-x}-\alpha(1-\xi)-i\epsilon}{\frac{1}{x}-\alpha\xi-i\epsilon}\psi'(\xi,x,r)\Big], \qquad (36)
\end{aligned}
$$

where $'$ means derivative with respect to x.

It is now straightforward to check that 't Hooft's solution $\psi_k(p_+,p_-,r)$ is indeed a solution also of this equation when α is set equal to α_k, for any value of η, in particular for $\eta = 1$, thanks to a precise cancellation of the contributions coming from the propagators ("virtual" insertions) against the extra term due to the modified form of the "potential" ("real" insertion). In other words the extra piece of the kernel at $\alpha = \alpha_k$ vanishes when acting on ψ_k as a perturbation. This phenomenon is analogous to the one occurring, with respect to the same extra term, in one loop perturbative four-dimensional calculations concerning Altarelli-Parisi [18] and Balitsky-Fadin-Kuraev-Lipatov [19] kernels. This analogy may have far-reaching consequences.

As a matter of fact, taking 't Hooft's equation into account, we get

$$
\begin{aligned}
&\Big[1-\frac{\eta g^2}{\pi M^2[1-\alpha x(1-x)-i\epsilon]^2}\Big((1-x)^2 + +[x^2[1+\frac{1-\alpha x(1-x)}{1-\alpha_k x(1-x)-i\epsilon}]\Big)\Big]\cdot \\
\cdot \quad &(\alpha_k-\alpha)\phi_k(x) = \frac{\eta g^2}{\pi M^2}\ \phi_k'(x)\ log\frac{1-\alpha_k x(1-x)-i\epsilon}{1-\alpha x(1-x)-i\epsilon}. \qquad (37)
\end{aligned}
$$

There are no corrections from a single insertion in the kernel to 't Hooft eigenvalues and eigenfunctions. We stress that this result does not depend on their detailed form, but only on their general properties. The ghosts which are responsible of the

causal behaviour of the ML propagator do not modify the bound state spectrum, as their "real" contribution cancels against the "virtual" one in propagators. Wu's equation for colorless bound states, although much more involved than the corresponding 't Hooft's one, might still apply. This is the heuristic lesson one learns from a single insertion in the kernel and is in agreement with the mentioned similar mechanism occurring in four-dimensional perturbative QCD.

Unfortunately this conclusion holds only at the level of a single insertion and may be a consequence of one loop unitarity which tightly relates 'real' to 'virtual' exchanges. Already when two insertions are taken into account, deviations are seen from 't Hooft spectrum [20]. This is not a surprise as Wu's equation is deeply different from 't Hooft's one and might describe the theory in a different phase (see for instance [21]).

Planarity plays a crucial role in both formulations; indeed the two equations sum exactly the same set of diagrams (the planar ones), which are thought to be the most important ones in the large N limit. The first lesson one learns is that planarity by itself is not sufficient to set up unambiguously a physical scenario.

Now there are good arguments [21] explaining why planarity should break down in the limit $m \to 0$. The same situation should occur when $m^2 < \frac{g_0^2}{\pi}$ which correspond to a 'strong' coupling situation, where we know that 't Hooft solution can no longer be trusted.

What about the 'weak' coupling regime? Should we believe that 't Hooft's picture describes correctly the physics in two dimensions, which in turn should be represented by planar diagrams, we would conclude that in 1+1 dimensions planarity is not a good approximation in the causal formulation of the theory. Indeed the results we obtain in the latter case are definitely different from 't Hooft ones. This is a very basic issue in our opinion, which definitely deserves further investigation. This is even more compelling should this situation persist in higher dimensions where causality is mandatory in order to obtain an acceptable formulation of the theory.

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